The Mathematics of Flat Fielding

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Neglecting dark current sources the noise in an image is expressed by the familiar noise equation

$$\text{Noise}_{\text{IMAGE}} = \sqrt{\text{Signal\_shot\_noise}^2 + \text{Fixed\_pattern\_noise}^2 + \text{Read\_noise}^2}$$

recall:

$$\text{Signal\_shot\_noise} = \sqrt{\text{Signal}}$$

and:

$$\text{Fixed\_pattern\_noise} = \text{PRNU} * \text{Signal}$$

When:

$$\text{Signal} > \frac{1}{\text{PRNU}^2}$$

the noise in the image becomes Fixed Pattern limited indicating the S/N is limited to 1/PRNU irrespective of signal level. This is undesirable and is the reason why Flat Fielding is applied to remove the Fixed Pattern Noise. This reduces the Noise equation to the theoretical best case:

$$\text{Noise}_{\text{IMAGE}} = \sqrt{\text{Signal} + \text{Read\_noise}^2}$$

The flat fielding operation consists of dividing, pixel by pixel, the raw image by a flat field image. The corrected ith pixel of an image that has been flat-fielded is expressed as:

$$S_{COR} = \mu_{FF} \frac{S_{RAW}}{S_{FF}} \tag{1}$$

$$S_{COR} = \text{corrected signal}$$

$$S_{FF} = \text{signal in flat field}$$

$$S_{RAW} = \text{signal in raw image}$$

$$\mu_{FF} = \text{average signal level in flat field}$$

We now seek the equation for the noise of the flat fielded image in terms of the signal level in the flat field and the raw image.

Since the noise of the corrected image is simply the square root of the variance of the image we need to calculate this variance.

The corrected image is a function of two variables, the raw signal and the flat field signal. To calculate the variance of a function of two variables, we use the propagation of errors formula:

$$\sigma_Q^2 = \sigma_x^2 \left(\frac{\partial Q}{\partial x}\right)^2 + \sigma_y^2 \left(\frac{\partial Q}{\partial y}\right)^2 + \sigma_{xy}^2 \left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial Q}{\partial y}\right)$$

If x and y are uncorrelated, $\sigma_{xy} = 0$

$$\sigma_Q^2 = \sigma_x^2 \left(\frac{\partial Q}{\partial x}\right)^2 + \sigma_y^2 \left(\frac{\partial Q}{\partial y}\right)^2 \qquad (2)$$

Because the noise of the raw image and flat field are uncorrelated we apply (2) to (1) and get:

$$\sigma_{\mathrm{COR}}^2 = \sigma_{\mathrm{FF-shot}}^2 \left(\frac{\partial \mathrm{S}_{\mathrm{COR}}}{\partial \mathrm{S}_{\mathrm{FF}}} \right)^2 + \sigma_{\mathrm{RAW-shot}}^2 \left(\frac{\partial \mathrm{S}_{\mathrm{COR}}}{\partial \mathrm{S}_{\mathrm{RAW}}} \right)^2$$

Including read noise for a practical system

$$\sigma_{\mathrm{COR}}^2 = \sigma_{\mathrm{FF-shot}}^2 \left(\frac{\partial \mathrm{S}_{\mathrm{COR}}}{\partial \mathrm{S}_{\mathrm{FF}}} \right)^2 + \sigma_{\mathrm{RAW-shot}}^2 \left(\frac{\partial \mathrm{S}_{\mathrm{COR}}}{\partial \mathrm{S}_{\mathrm{RAW}}} \right)^2 + \sigma_{\mathrm{READ}}^2 \qquad (3)$$

Substituting (1) into (3) we get

$$\sigma_{\mathrm{COR}}^{2} = \sigma_{\mathrm{FF\text{-}shot}}^{2}\left(\frac{\partial\left(\mu_{\mathrm{FF}}\dfrac{\mathrm{S}_{\mathrm{RAW}}}{\mathrm{S}_{\mathrm{FF}}}\right)}{\partial \mathrm{S}_{\mathrm{FF}}}\right)^{2} + \sigma_{\mathrm{RAW\text{-}shot}}^{2}\left(\frac{\partial\left(\mu_{\mathrm{FF}}\dfrac{\mathrm{S}_{\mathrm{RAW}}}{\mathrm{S}_{\mathrm{FF}}}\right)}{\partial \mathrm{S}_{\mathrm{RAW}}}\right)^{2} + \sigma_{\mathrm{READ}}^{2} \qquad (4)$$

Performing the differentiation, equation (4) reduces to:

$$\sigma_{\mathrm{COR}}^{2} = \sigma_{\mathrm{FF-shot}}^{2}\left(-\mu_{\mathrm{FF}}\frac{\mathrm{S}_{\mathrm{RAW}}}{\mathrm{S}^{2}_{\mathrm{FF}}}\right)^{2} + \sigma_{\mathrm{RAW-shot}}^{2}\left(\frac{\mu_{\mathrm{FF}}}{\mathrm{S}_{\mathrm{FF}}}\right)^{2} + \sigma_{\mathrm{READ}}^{2} \qquad (5)$$

noting $\mu_{\mathrm{FF}} = \mathrm{S}_{\mathrm{FF}}$

equation (5) further reduces to

$$\sigma^2_{\mathrm{COR}} = \sigma^2{}_{\text{FF-shot}} \frac{S^2_{\mathrm{RAW}}}{S^2_{\mathrm{FF}}} + \sigma^2{}_{\text{RAW-shot}} + \sigma^2_{\mathrm{READ}} \qquad (6)$$

since $\sigma^2{}_{\text{FF-shot}} = S_{\mathrm{FF}}$

and $\sigma^2{}_{\text{RAW-shot}} = S_{\mathrm{RAW}}$

equation (6) simplifies to

$$\sigma^2_{\mathrm{COR}} = S_{\mathrm{RAW}}\left(1 + \frac{S_{\mathrm{RAW}}}{S_{\mathrm{FF}}}\right) + \sigma^2_{\mathrm{READ}} \qquad (7)$$

so long as $S_{FF} >> S_{RAW}$ equation (7) can be deemed to be

$$\sigma_{COR}^2 = S_{RAW} + \sigma_{READ}^2$$

which is shot noise limited when $S_{RAW} > \sigma_{READ}^2$

indicating the Fixed Pattern Noise is completely removed. Unfortunately with a finite well depth the inequality cannot always be guaranteed when using a single flat field frame to calibrate a raw image containing a high signal level. A solution can be found by combining N_{FF} frames of signal level Q_{FF}

$$\sigma_{COR}^2 = S_{RAW}\left(1 + \frac{S_{RAW}}{N_{FF}Q_{FF}}\right) + \sigma_{READ}^2 \qquad (8)$$

Since any arbitrary number of flat field images can be combined together, it is a simple matter to guarantee $N_{FF}Q_{FF} >> S_{RAW}$ by selecting an appropriate value of N_{FF} such that (8) simplifies to

$$\sigma^2_{COR} = S_{RAW} + \sigma^2_{READ} \tag{9}$$

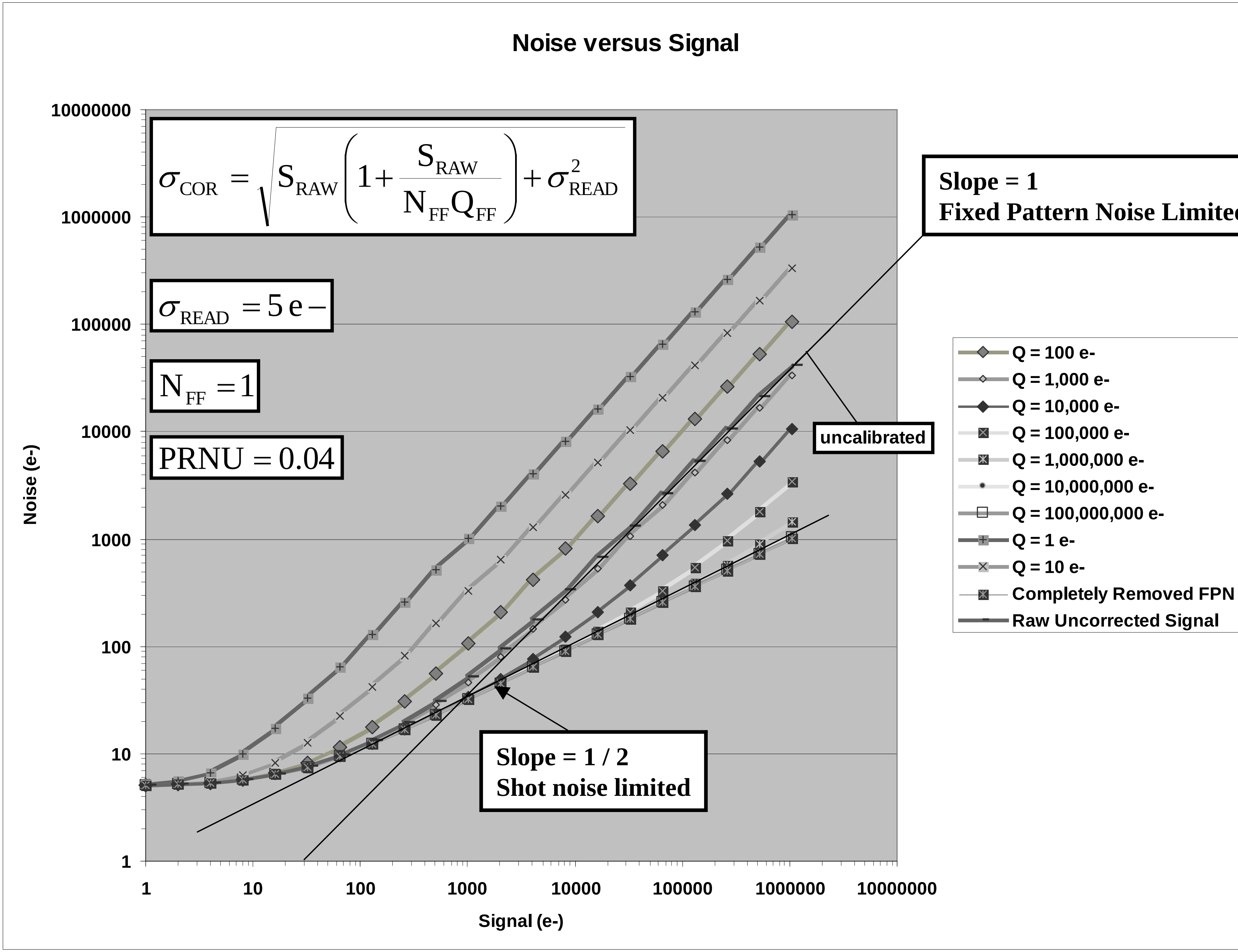
Taking the square root of each side (9) transforms to our desired noise equation

$$\mathrm{Noise}_{IMAGE} = \sqrt{\mathrm{Signal} + \mathrm{Read\_noise}^2}$$

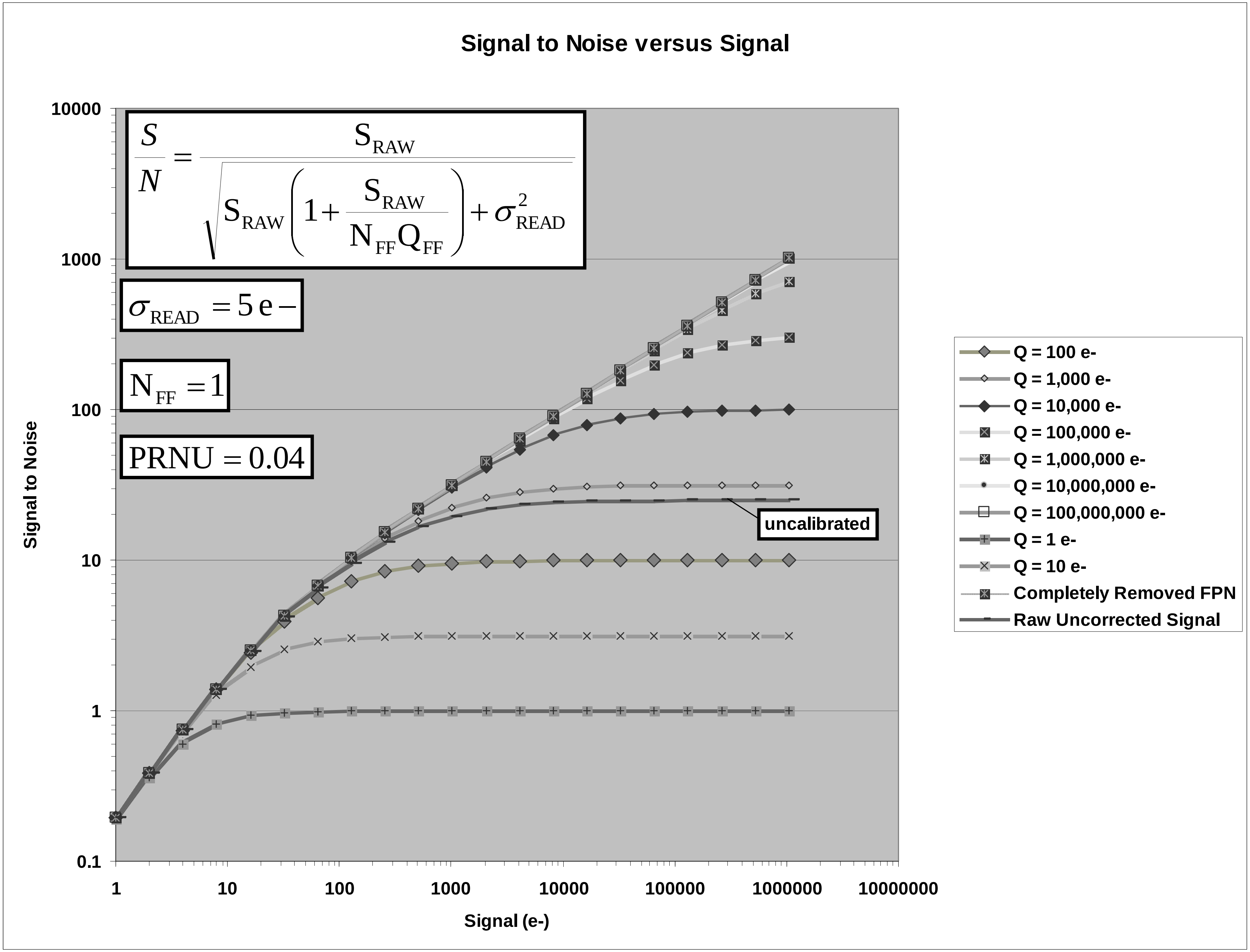
Illustrating the concepts are the following Flat Field Photon Transfer Curves (FFPTC). The first plots the noise versus signal for a raw image and the same image calibrated with a collection of flat field images of varying signal intensity. For low signal values in the flat field, the noise of the corrected image is increased, defeating the purpose of flat fielding.

The second uses the same data as the first but plots the S/N versus signal instead. Again for low signal values in the flat field, the noise of the corrected image is increased, defeating the purpose of flat fielding.

To determine the optimum signal level in the flat field images an FFPTC analysis is performed.



Noise versus Signal
$$\sigma_{COR} = \sqrt{S_{RAW}\left(1+\frac{S_{RAW}}{N_{FF}Q_{FF}}\right)+\sigma_{READ}^2}$$
$\sigma_{READ} = 5\,e-$
$N_{FF} = 1$
PRNU = 0.04
Slope = 1
Fixed Pattern Noise Limited
uncalibrated
Slope = 1 / 2
Shot noise limited
Q = 100 e-
Q = 1,000 e-
Q = 10,000 e-
Q = 100,000 e-
Q = 1,000,000 e-
Q = 10,000,000 e-
Q = 100,000,000 e-
Q = 1 e-
Q = 10 e-
Completely Removed FPN
Raw Uncorrected Signal
Noise (e-)
Signal (e-)
1
10
100
1000
10000
100000
1000000
10000000



Signal to Noise versus Signal
$$\frac{S}{N} = \frac{S_{RAW}}{\sqrt{S_{RAW}\left(1+\frac{S_{RAW}}{N_{FF}Q_{FF}}\right)+\sigma_{READ}^2}}$$
$\sigma_{READ} = 5\,e-$
$N_{FF} = 1$
$PRNU = 0.04$
uncalibrated
Signal to Noise
Signal (e-)
10000
1000
100
10
1
0.1
1
10
100
1000
10000
100000
1000000
10000000
Q = 100 e-
Q = 1,000 e-
Q = 10,000 e-
Q = 100,000 e-
Q = 1,000,000 e-
Q = 10,000,000 e-
Q = 100,000,000 e-
Q = 1 e-
Q = 10 e-
Completely Removed FPN
Raw Uncorrected Signal