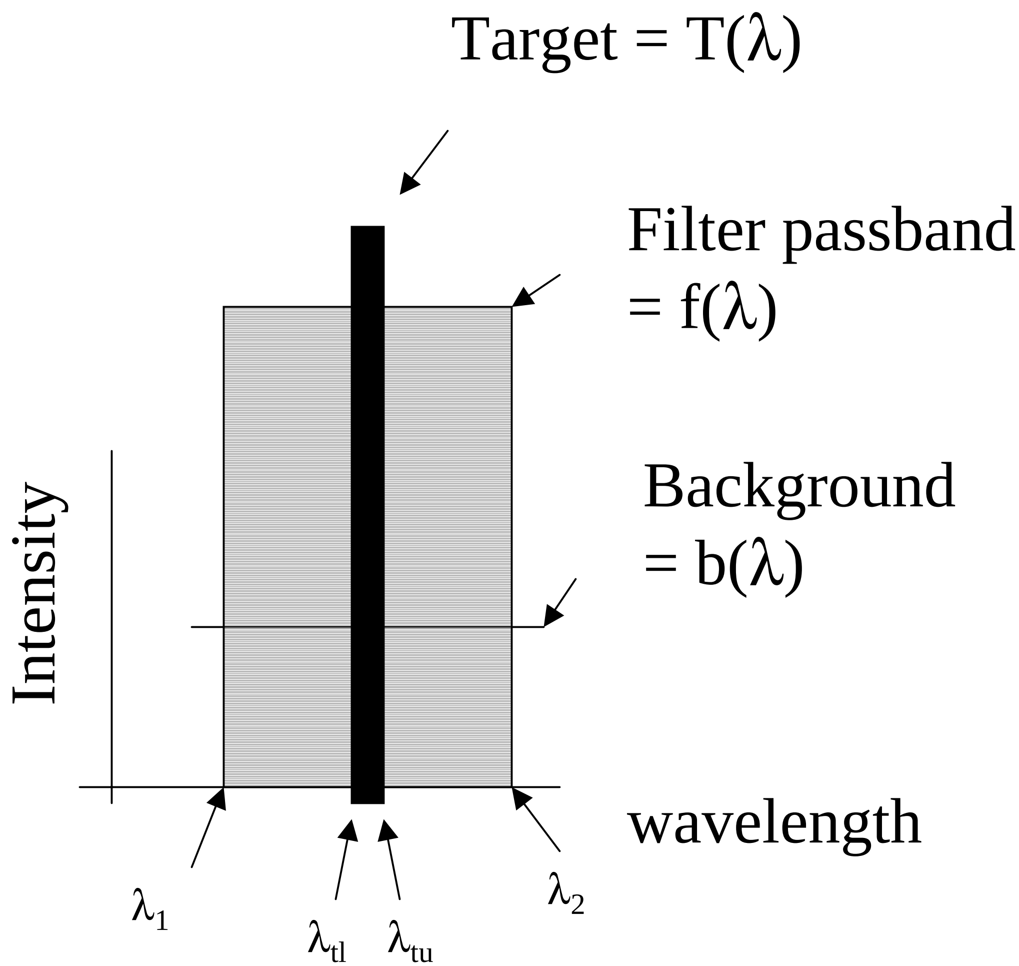
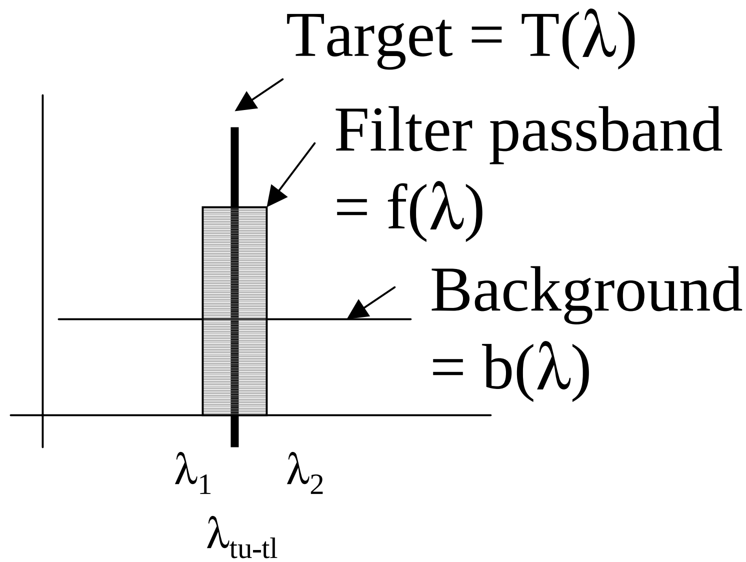
Spectral Discrimination of Targets Using Filters

Goal: develop an analytical expression for the signal to noise ratio as referred to the filter for using narrowband filters to spectrally discriminate a target in the presence of in-band noise





Assume:

1) Filter passband, $f(\lambda)$, is rectangle defined:

$$
\begin{aligned}
f(\lambda) &= 0; && 0 \leq \lambda < \lambda_1 \\
&= f_{thru}; && \lambda_1 \leq \lambda \leq \lambda_2 \\
&= 0; && \lambda_2 < \lambda < \text{infinity}
\end{aligned}
$$

2) $\lambda_1 < \lambda_{tl} < \lambda_{tu} < \lambda_2$

3) Background level is constant over filter bandwidth[1]

[1] A notable exception is the [NII] doublet surrounding Halpha. One line is at 2nm separation, another is at 1.5nm separation. Some nebulae have significant [NII] emission and that is defined as noise in an Halpha image

$$\text{Signal\_to\_noise\_ratio} = \text{SNR}^{(2,3)} = \frac{\text{Signal}}{\text{Noise}} = \frac{\int_0^\infty f(\lambda)s(\lambda)d\lambda}{\left(\int_0^\infty f^2(\lambda)b^2(\lambda)d\lambda + N^2\right)^{1/2}}$$

where:

$s(\lambda)$ = signal(λ)
$f(\lambda)$ = filter response(λ)
$b(\lambda)$ = background(λ)
N = system read noise

$s(\lambda) = T(\lambda) - b(\lambda)$
$b(\lambda) = k$
$T(\lambda) = T_{max} * (u(\lambda - \lambda_{tl}) - u(\lambda - \lambda_{tu}))$
$f(\lambda) = f_{thru} * (u(\lambda - \lambda_1) - u(\lambda - \lambda_2))$

where:

u(x) = unit step function (aka Heaviside step function)

[2] W. D. Montgomery, “Some Consequences of Sampling in FLIR Systems”, Institute for Defense Analyses, Arlington, Va. , Research Paper P-543, September 1969
[3] W. D. Davenport and W. L. Root, “Random Signals and Noise” McGraw-Hill, New York, N.Y., 1958.

$$Signal = \int_0^\infty f(\lambda) * s(\lambda) d\lambda$$

$$= \int_0^\infty f_{thru} * \big(u(\lambda - \lambda_1) - u(\lambda - \lambda_2)\big) * \big(T_{\max} * (u(\lambda - \lambda_{tl}) - u(\lambda - \lambda_{tu})) - k\big) d\lambda$$

To evaluate this integral, recall:

1) $u(\lambda - \lambda_1) =$ 0 for $0 </= \lambda < \lambda_1$
$=$ 1 for $\lambda_1 </= \lambda$

Think of this as a “switch”
That turns on at $\lambda = \lambda_1$

2) An integral over the interval 0 to infinity can be decomposed into the sum of integrals over smaller intervals

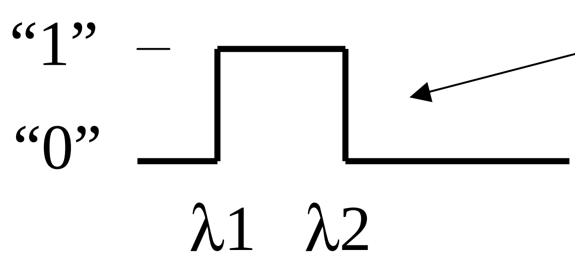
3) Any interval where the quantity under the integral sign is zero can be ignored

$$Signal = \int_0^\infty f_{thru} * \left(u(\lambda-\lambda_1)-u(\lambda-\lambda_2)\right) * \left(T_{\max} * (u(\lambda-\lambda_{tl})-u(\lambda-\lambda_{tu}))-k\right)d\lambda$$

OK, let’s do it

$$Signal = \int_0^\infty f_{thru} * \boxed{\left(u(\lambda-\lambda_1)-u(\lambda-\lambda_2)\right)} * \left(T_{\max} * (u(\lambda-\lambda_{tl})-u(\lambda-\lambda_{tu}))-k\right)d\lambda$$

This value is “0” except for the interval: $\lambda_1 - \lambda_2$



So we can reduce the integral to the interval from $\lambda_1 - \lambda_2$ only

$$Signal = \int_{\lambda_1}^{\lambda_2} f_{thru} * \left(u(\lambda-\lambda_1)-u(\lambda-\lambda_2)\right) * \left(T_{\max} * (u(\lambda-\lambda_{tl})-u(\lambda-\lambda_{tu}))-k\right)d\lambda$$

$$Signal = \int_{\lambda_1}^{\lambda_2} f_{thru} * \left(u(\lambda-\lambda_1)-u(\lambda-\lambda_2)\right) * \left(T_{\max} * (u(\lambda-\lambda tl)-u(\lambda-\lambda tu))-k\right) d\lambda$$

Within the interval, $\lambda_1 - \lambda_2$, the quantity:

$(u(\lambda - \lambda_1) - u(\lambda - \lambda_2)) = 1$
And outside that interval it is equal to zero

So the integral simplifies to:

$$Signal = \int_{\lambda_1}^{\lambda_2} f_{thru} * \left(T_{\max} * (u(\lambda-\lambda tl)-u(\lambda-\lambda tu))-k\right) d\lambda$$

If we apply similar logic as above to the next term of the integral that contains the Heaviside step function, and then stipulate that the filter passband will be wider than the target's bandwidth we can further reduce the integral to:

$$Signal = \int_{\lambda_1}^{\lambda tl} f_{thru} * (-k) d\lambda + \int_{\lambda tl}^{\lambda tu} f_{thru} * (T\max - k) d\lambda + \int_{\lambda tu}^{\lambda_2} f_{thru} * (-k) d\lambda$$

$$Signal = \int_{\lambda_1}^{\lambda_{tl}} f_{thru} * \left(-k\right) d\lambda + \int_{\lambda_{tl}}^{\lambda_{tu}} f_{thru} * \left(T_{\max} - k\right) d\lambda + \int_{\lambda_{tu}}^{\lambda_2} f_{thru} * \left(-k\right) d\lambda$$

Rearranging terms and combining like quantities we get:

$$Signal = \int_{\lambda_1}^{\lambda_2} f_{thru} * \left(-k\right) d\lambda + \int_{\lambda_{tl}}^{\lambda_{tu}} f_{thru} * T_{\max}\, d\lambda$$

Noting that each quantity under the integral is a constant and recalling that the definite integral of a constant over a range is equal to the constant multiplied by the range we get;

$$Signal = f_{thru} * \left(T_{\max} * (\lambda_{tu} - \lambda_{tl}) - k * (\lambda_2 - \lambda_1)\right)$$

or in plain words:

Signal/pixel = filter_throughput*[target_strength*target_bandwidth – background_strength*filter_bandwidth]

$$Noise = \left(\int_0^\infty f^2(\lambda) b^2(\lambda) d\lambda + N^2\right)^{1/2}$$

Due to the rectangular filter passband and the constant background level, the noise equation for this case transforms to:

$$Noise = \left(\int_0^\infty f_{thru}{}^2 (u(\lambda - \lambda_1) - u(\lambda - \lambda_2))^2 * k^2) d\lambda + N^2\right)^{1/2}$$

where:

$f(\lambda)$ = filter response(λ) = f_{thru} * $(u(\lambda - \lambda_1) - u(\lambda - \lambda_2))$

$b(\lambda)$ = background(λ) = k

and:

u(x) = unit step function (aka Heaviside step function)

$$Noise = \left(\int_0^\infty f_{thru}{}^2 (u(\lambda - \lambda_1) - u(\lambda - \lambda_2))^2 * k^2 \, d\lambda + N^2 \right)^{1/2}$$

where:

$f(\lambda)$ = filter response(λ) = $f_{thru} * (u(\lambda - \lambda_1) - u(\lambda - \lambda_2))$

$b(\lambda)$ = background(λ) = k (over the filter passband region)

N = system read noise

and:

u(x) = unit step function (aka Heaviside step function)

Again, observing the characteristics of the step function and the properties of integrals the solution becomes:

$$Noise = \left(f_{thru}{}^2 k^2 * (\lambda_2 - \lambda_1) + N^2 \right)^{1/2}$$

So the signal to noise relationship for a rectangular bandpass filter with a rectangular spectral distribution of the target signal and a constant background level over the passband of the filter is therefore:

$$\text{SNR/pixel=} \quad \frac{f_{thru} * \left(T_{\max} * (\lambda_{tu} - \lambda_{tl}) - k * (\lambda_2 - \lambda_1)\right)}{\left(f_{thru}{}^2 k^2 * (\lambda_2 - \lambda_1) + N^2\right)^{1/2}}$$

$$\text{SNR/Pixel=} \quad \frac{\text{Filter\_throughput*(target\_strength*target\_bandwidth – background\_strength*filter\_bandwidth)}}{\text{[filter\_throughput}^2\text{*background\_level}^2\text{*(filter\_bandwidth) + read\_noise}^2\text{]}^{1/2}}$$

Application of the Theory

$$\text{SNR} = \frac{\int_0^\infty f(\lambda)s(\lambda)d\lambda}{\left(\int_0^\infty f^2(\lambda)b^2(\lambda)d\lambda + N^2\right)^{1/2}}$$

Example 1:

$$\int_0^\infty f(\lambda)s(\lambda)d\lambda = 311.3(e-)$$

$$\int_0^\infty f(\lambda)b(\lambda)d\lambda = 50.58(e-)$$

Read Noise = 10(e-)
SNR = 6.0377

	CS Ha (3nm)	AD Ha (6nm)	CS O3(3nm)	AD O3 (6nm)	CS S2 (3nm)	AD S2 (6nm)
Target e-	311.3	461.67	81.08	169.25	83.25	135
Bkgd e-	50.58	84.83	58.25	136.36	48.91	92.08
read noise e-	10	10	10	10	10	10
SNR	6.037736446	5.404871901	1.37186243	1.23787554	1.66760748	1.457546302

Example 2:

$$\int_0^\infty f(\lambda)s(\lambda)d\lambda = 461.3(e-)$$

$$\int_0^\infty f(\lambda)b(\lambda)d\lambda = 84.83(e-)$$

Read Noise = 10(e-)
SNR = 5.40(e-)